

Gröbner basis and singular locus of Lauricella's hypergeometric differential equations

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1 Introduction

Lauricella's hypergeometric series F_A, F_B, F_C are defined by

$$F_A(a, b_1, \dots, b_m, c_1, \dots, c_m; x_1, \dots, x_m) = \sum_{n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}} \frac{(a)_{n_1+\dots+n_m} (b_1)_{n_1} \cdots (b_m)_{n_m}}{(c_1)_{n_1} \cdots (c_m)_{n_m} (1)_{n_1} \cdots (1)_{n_m}} x_1^{n_1} \cdots x_m^{n_m},$$

$$F_B(a_1, \dots, a_m, b_1, \dots, b_m, c; x_1, \dots, x_m) = \sum_{n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}} \frac{(a_1)_{n_1} \cdots (a_m)_{n_m} (b_1)_{n_1} \cdots (b_m)_{n_m}}{(c)_{n_1+\dots+n_m} (1)_{n_1} \cdots (1)_{n_m}} x_1^{n_1} \cdots x_m^{n_m},$$

$$F_C(a, b, c_1, \dots, c_m; x_1, \dots, x_m) = \sum_{n_1, \dots, n_m \in \mathbb{Z}_{\geq 0}} \frac{(a)_{n_1+\dots+n_m} (b)_{n_1+\dots+n_m}}{(c_1)_{n_1} \cdots (c_m)_{n_m} (1)_{n_1} \cdots (1)_{n_m}} x_1^{n_1} \cdots x_m^{n_m}.$$

Here, a, b, c, a_i, b_i, c_i ($i = 1, \dots, m$) are parameters and $c, c_i \notin \mathbb{Z}_{\leq 0}$. Lauricella's series F_A, F_B, F_C satisfy the following systems of differential equations respectively

$$\ell_i^A \cdot F_A = 0, \ell_i^A = \theta_i(\theta_i + c_i - 1) - x_i(\theta_1 + \cdots + \theta_m + a)(\theta_i + b_i) \quad (i = 1, \dots, m),$$

$$\ell_i^B \cdot F_B = 0, \ell_i^B = \theta_i(\theta_1 + \cdots + \theta_m + c - 1) - x_i(\theta_i + a_i)(\theta_i + b_i) \quad (i = 1, \dots, m),$$

$$\ell_i^C \cdot F_C = 0, \ell_i^C = \theta_i(\theta_i + c_i - 1) - x_i(\theta_1 + \cdots + \theta_m + a)(\theta_1 + \cdots + \theta_m + b) \quad (i = 1, \dots, m).$$

Here, $\partial_i = \frac{\partial}{\partial x_i}$ is the differential operator for x_i and $\theta_i = x_i \partial_i$ is Euler operator for x_i .

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Let $D = \mathbb{C}[x_1, \dots, x_m] \langle \partial_1, \dots, \partial_m \rangle$ be the ring of differential operators with polynomial coefficients and $\mathcal{D} = \mathbb{C}\{x_1, \dots, x_m\} \langle \partial_1, \dots, \partial_m \rangle$ be the ring of differential operators with convergent power series coefficients. We define D ideals

$$\begin{aligned} I_A(m) &= D \cdot \{\ell_i^A \mid i = 1, \dots, m\}, \\ I_B(m) &= D \cdot \{\ell_i^B \mid i = 1, \dots, m\}, \\ I_C(m) &= D \cdot \{\ell_i^C \mid i = 1, \dots, m\}, \end{aligned}$$

and \mathcal{D} ideals

$$\mathcal{I}_A(m) = \mathcal{D} \cdot \{\ell_i^A \mid i = 1, \dots, m\}, \quad \mathcal{I}_C(m) = \mathcal{D} \cdot \{\ell_i^C \mid i = 1, \dots, m\}.$$

We note that these ideals can be defined without any condition on parameters. We will call these ideals Lauricella's systems of differential equations.

In this paper, we obtain Gröbner bases for these ideals $I_B(m), \mathcal{I}_A(m), \mathcal{I}_C(m)$ with respect to some monomial orders without any condition on parameters. By utilizing these Gröbner bases and by Oaku's celebrating result on Gröbner basis, characteristic varieties and singular locus [6], [7], we will determine the singular locus of Lauricella's system of differential equations $I_t(m)$ where t is A or B or C .

The singular locus under some conditions on parameters is known for these equations (see the survey by Matsumoto [5]). However, the singular locus without any condition on parameters has not been known. We note that Hattori and Takayama [4] determined the singular locus of the system $I_C(m)$ recently without any assumption on parameters by utilizing Gröbner basis, syzygies and cohomological solutions. Our method also utilizes Gröbner basis, but it is simpler by considering Gröbner bases in the ring \mathcal{D} and can be applied to other Lauricella's systems. We will use notations of [9] throughout this paper.

2 Gröbner basis for Lauricella's hypergeometric differential equations

We derive a Gröbner basis for the D ideal $I_B(m)$.

Theorem 1. *Let ξ_i be a commutative variable corresponding to ∂_i . We define a term order $<_{(0,1)}$ as follows. The relation*

$$x_1^{\alpha_1} \dots x_m^{\alpha_m} \xi_1^{\beta_1} \dots \xi_m^{\beta_m} <_{(0,1)} x_1^{\alpha'_1} \dots x_m^{\alpha'_m} \xi_1^{\beta'_1} \dots \xi_m^{\beta'_m}$$

holds if and only if one of the following cases holds:

1. $\beta_1 + \dots + \beta_m < \beta'_1 + \dots + \beta'_m$
2. $\beta_1 + \dots + \beta_m = \beta'_1 + \dots + \beta'_m$ and $\alpha_1 + \dots + \alpha_m < \alpha'_1 + \dots + \alpha'_m$
3. $\beta_1 + \dots + \beta_m = \beta'_1 + \dots + \beta'_m$ and $\alpha_1 + \dots + \alpha_m = \alpha'_1 + \dots + \alpha'_m$ and $x_1^{\alpha_1} \dots x_m^{\alpha_m} \xi_1^{\beta_1} \dots \xi_m^{\beta_m} <' x_1^{\alpha'_1} \dots x_m^{\alpha'_m} \xi_1^{\beta'_1} \dots \xi_m^{\beta'_m}$. Here, $<'$ is a term order such as the lexicographic order.

Then, the set $\{\ell_1^B, \dots, \ell_m^B\}$ is a Gröbner basis for $I_B(m)$ with respect to $<_{(0,1)}$.

In order to prove the theorem, we use the Buchberger's criterion. In other words, we prove that any S-pair is reduced to 0. We will use the following lemma to simplify S-pairs.

Lemma 2. *Let $P, Q \in D$ and $<$ be a term order on D . If the initial terms $\text{in}_<(P)$ and $\text{in}_<(Q)$ are relatively prime, the S-pair of P and Q $S_<(P, Q)$ is reduced to the commutator $-[P, Q]$.*

Proof. We may assume that the coefficients of the initial terms of P and Q are 1 without loss of generality. Since the initial terms $\text{in}_<(P)$ and $\text{in}_<(Q)$ are relatively prime, we have

$$\begin{aligned} S_<(P, Q) &= (\text{in}_<(Q)(x, \partial))P - (\text{in}_<(P)(x, \partial))Q \\ &= (Q - \text{rest}_<(Q))P - (P - \text{rest}_<(P))Q \\ &= -\text{rest}_<(Q)P + \text{rest}_<(P)Q + QP - PQ \\ &= -\text{rest}_<(Q)P + \text{rest}_<(P)Q - [P, Q]. \end{aligned}$$

When $\text{in}_<(P) = x_1^{\alpha_1} \dots x_m^{\alpha_m} \xi_1^{\beta_1} \dots \xi_m^{\beta_m}$, we define

$$\text{in}_<(P)(x, \partial) = x_1^{\alpha_1} \dots x_m^{\alpha_m} \partial_1^{\beta_1} \dots \partial_m^{\beta_m}$$

and $\text{rest}_<(P) = P - \text{in}_<(P)(x, \partial)$. The S-pair $S_<(P, Q)$ is reduced to the commutator $-[P, Q]$ by P and Q . \square

Proof. (of Theorem 1) We need to show that the S-pair of ℓ_i^B, ℓ_j^B ($1 \leq i < j \leq m$) is reduced to 0. Since the initial terms $\text{in}_{<_{(0,1)}}(\ell_i^B) = x_i^3 \xi_i^2$ and $\text{in}_{<_{(0,1)}}(\ell_j^B) = x_j^3 \xi_j^2$ are relatively prime, we can use Lemma 2. The commutator of ℓ_i^B and ℓ_j^B is

$$\begin{aligned} [\ell_i^B, \ell_j^B] &= \ell_i^B \ell_j^B - \ell_j^B \ell_i^B \\ &= x_i(\theta_i + a_i)(\theta_i + b_i)\theta_j - x_j(\theta_j + a_j)(\theta_j + b_j)\theta_i \\ &\xrightarrow[\ell_i^B, \ell_j^B]{*} \theta_i(\theta_1 + \dots + \theta_m + c - 1)\theta_j - \theta_j(\theta_1 + \dots + \theta_m + c - 1)\theta_i = 0, \end{aligned}$$

where $\xrightarrow[\ell_i^B, \ell_j^B]{*}$ means the reduction by ℓ_i^B and ℓ_j^B . Since the commutator is reduced to 0, the S-pair $S_{<_{(0,1)}}(\ell_i^B, \ell_j^B)$ is reduced to 0 by Lemma 2. By the Buchberger's criterion, the set $\{\ell_1^B, \dots, \ell_m^B\}$ is a Gröbner basis with respect to $<_{(0,1)}$. \square

Remark 3. *In Theorem 1, we obtain a Gröbner basis of $I_B(m)$ with respect to the term order $<_{(0,1)}$. We are interested in the set of term orders for which the set of generators $\{\ell_1^B, \dots, \ell_m^B\}$ is a Gröbner basis. Let us determine the weight vector w and the tie-breaker order such that the term order $<_w$ satisfies*

$\text{in}_{<_w}(\ell_i^B) = x_i^3 \xi_i^2$. This condition yields the condition that the weight vector $w = (w_1, \dots, w_m, w_{m+1}, \dots, w_{2m}) \in (\mathbb{R}_{\geq 0})^{2m}$ satisfies

$$w_i > 0, w_{m+i} \geq 0, 2w_i - w_k + w_{m+i} - w_{m+k} > 0 \quad (1 \leq k \leq m \text{ and } k \neq i) \quad (1)$$

for $i = 1, \dots, m$. We define

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m} \xi_1^{\beta_1} \cdots \xi_m^{\beta_m} <_w x_1^{\alpha'_1} \cdots x_m^{\alpha'_m} \xi_1^{\beta'_1} \cdots \xi_m^{\beta'_m}$$

if and only if one of the following cases holds:

1. $w_1 \alpha_1 + \cdots + w_m \alpha_m + w_{m+1} \beta_1 + \cdots + w_{2m} \beta_m < w_1 \alpha'_1 + \cdots + w_m \alpha'_m + w_{m+1} \beta'_1 + \cdots + w_{2m} \beta'_m$
2. $w_1 \alpha_1 + \cdots + w_m \alpha_m + w_{m+1} \beta_1 + \cdots + w_{2m} \beta_m = w_1 \alpha'_1 + \cdots + w_m \alpha'_m + w_{m+1} \beta'_1 + \cdots + w_{2m} \beta'_m$ and $x_1^{\alpha_1} \cdots x_m^{\alpha_m} \xi_1^{\beta_1} \cdots \xi_m^{\beta_m} < x_1^{\alpha'_1} \cdots x_m^{\alpha'_m} \xi_1^{\beta'_1} \cdots \xi_m^{\beta'_m}$. Here, $<$ is a term order such as the lexicographic order.

We can prove that the set of generators $\{\ell_1^B, \dots, \ell_m^B\}$ is a Gröbner basis with respect to the term order $<_w$. We note that the weight vector $(0, \dots, 0, 1, \dots, 1)$ lies in the closure of the cone (1) in the weight space. The set of generators $\{\ell_1^B, \dots, \ell_m^B\}$ is a Gröbner basis with respect to the term order defined by the weight vector $(0, \dots, 0, 1, \dots, 1)$ and the tie-breaker $<_w$.

Next, We derive a Gröbner basis for \mathcal{D} ideal $\mathcal{I}_A(m)$.

Theorem 4. We define a monomial order on $<_{(0,1)'}'$ on \mathcal{D} as follows. The relation

$$x_1^{\alpha_1} \cdots x_m^{\alpha_m} \xi_1^{\beta_1} \cdots \xi_m^{\beta_m} <_{(0,1)'}' x_1^{\alpha'_1} \cdots x_m^{\alpha'_m} \xi_1^{\beta'_1} \cdots \xi_m^{\beta'_m}$$

holds if and only if one of the following case holds:

1. $\beta_1 + \cdots + \beta_m < \beta'_1 + \cdots + \beta'_m$
2. $\beta_1 + \cdots + \beta_m = \beta'_1 + \cdots + \beta'_m$ and $\alpha_1 + \cdots + \alpha_m > \alpha'_1 + \cdots + \alpha'_m$
3. $\beta_1 + \cdots + \beta_m = \beta'_1 + \cdots + \beta'_m$ and $\alpha_1 + \cdots + \alpha_m = \alpha'_1 + \cdots + \alpha'_m$ and $x_1^{\alpha_1} \cdots x_m^{\alpha_m} \xi_1^{\beta_1} \cdots \xi_m^{\beta_m} <' x_1^{\alpha'_1} \cdots x_m^{\alpha'_m} \xi_1^{\beta'_1} \cdots \xi_m^{\beta'_m}$. Here, $<'$ is a term order such as the lexicographic order.

Then, the set $\{\ell_1^A, \dots, \ell_m^A\}$ is a Gröbner basis for \mathcal{D} ideal $\mathcal{I}_A(m)$ with respect to $<_{(0,1)'}'$.

Before giving a proof, we note that for the monomial order $<_{(0,1)'}'$ in the ring \mathcal{D} , an analogous lemma as Lemma 2 also holds.

Proof. We need to prove the S -pair of ℓ_i^A and ℓ_j^A ($1 \leq i < j \leq m$) is reduced to 0. Since the initial terms $\text{in}_{<_{(0,1)'}'}(\ell_i^A) = x_i^2 \xi_i^2$ and $\text{in}_{<_{(0,1)'}'}(\ell_j^A) = x_j^2 \xi_j^2$ are relatively prime, we can use the analogous lemma as Lemma 2. The commutator $[\ell_i^A, \ell_j^A] = 0$. The S -pair $S'_{<_{(0,1)'}'}(\ell_i^A, \ell_j^A)$ is reduced to 0. By the Buchberger's criterion with respect to a monomial order $<_{(0,1)'}'$ in \mathcal{D} [1], [6, Th 1.4.], the set $\{\ell_1^A, \dots, \ell_m^A\}$ is a Gröbner basis. \square

For \mathcal{D} ideal $\mathcal{I}_C(m)$, we can also derive a Gröbner basis analogously.

Theorem 5. *The set $\{\ell_1^C, \dots, \ell_m^C\}$ is a Gröbner basis for \mathcal{D} ideal $\mathcal{I}_C(m)$ with respect to $<_{(\mathbf{0}, \mathbf{1})}'$.*

Remark 6. *We could not derive Gröbner bases for D ideals $I_A(m), I_C(m)$ and the D ideal for F_D with respect to the term order $<_{(\mathbf{0}, \mathbf{1})}$. Gröbner bases for these ideals seem to be more complicated from computer experiments. This is why we discuss on a Gröbner basis in the ring \mathcal{D} to study the singular locus.*

3 Singular locus of Lauricella's system $I_A(m), I_B(m)$

Hattori and Takayama [4] determined the singular locus of Lauricella's system $I_C(m)$ by using Gröbner basis, syzygies and cohomological solutions. We determine the singular locus of Lauricella's system $I_A(m), I_B(m)$ by using the obtained Gröbner bases.

3.1 Singular locus of Lauricella's system $I_B(m)$

We compute singular locus of Lauricella's system $I_B(m)$. By Theorem 1, a Gröbner basis of $I_B(m)$ with respect to $<_{(\mathbf{0}, \mathbf{1})}$ is $\{\ell_1^B, \dots, \ell_m^B\}$. We define a weight vector $(\mathbf{0}, \mathbf{1}) = (0, \dots, 0, 1, \dots, 1) \in \mathbb{Z}^{2m}$. In other words, we set that the weight of x_i is 0 and that of ξ_i is 1. We define the $(\mathbf{0}, \mathbf{1})$ initial form $\text{in}_{(\mathbf{0}, \mathbf{1})}(P)$ of the differential operator P by the sum of the terms in P which has the highest $(\mathbf{0}, \mathbf{1})$ weight. In other words, when $P = \sum_{\alpha, \beta \in (\mathbb{Z}_{\geq 0})^m} c_{\alpha, \beta} x^\alpha \partial^\beta \in D$, the $(\mathbf{0}, \mathbf{1})$ initial form of P is

$$\text{in}_{(\mathbf{0}, \mathbf{1})}(P) = \sum_{(\mathbf{0}, \mathbf{1}) \cdot (\alpha, \beta) \text{ is maximum in } P} c_{\alpha, \beta} x^\alpha \xi^\beta.$$

For a D ideal I , the $(\mathbf{0}, \mathbf{1})$ initial form ideal is defined by the $\mathbb{C}[x, \xi]$ ideal

$$\text{in}_{(\mathbf{0}, \mathbf{1})}(I) = \langle \text{in}_{(\mathbf{0}, \mathbf{1})}(P) \mid P \in I \rangle.$$

By the property of Gröbner basis with respect to $<_{(\mathbf{0}, \mathbf{1})}$, the $(\mathbf{0}, \mathbf{1})$ initial form ideal $\text{in}_{(\mathbf{0}, \mathbf{1})}(I_B(m))$ are generated by $\text{in}_{(\mathbf{0}, \mathbf{1})}(\ell_1^B), \dots, \text{in}_{(\mathbf{0}, \mathbf{1})}(\ell_m^B)$. The $(\mathbf{0}, \mathbf{1})$ initial form of ℓ_i^B is

$$\text{in}_{(\mathbf{0}, \mathbf{1})}(\ell_i^B) = x_i \xi_i \left(x_i (1 - x_i) \xi_i + \sum_{1 \leq j \leq m, j \neq i} x_j \xi_j \right).$$

We denote the initial form by L_i^B . By the Oaku's result [7, Proposition 1 in Section 2] and Theorem 1, we have the following proposition.

Proposition 7. *The characteristic variety for D ideal $I_B(m)$ is $\text{Ch}(I_B(m)) = \mathbf{V}(L_1^B, \dots, L_m^B)$.*

The singular locus of $I_B(m)$ is defined by

$$\text{Sing}(I_B(m)) = \pi(\text{Ch}(I_B(m)) \setminus \{\xi_1 = \dots = \xi_m = 0\}).$$

Here, π is the projection $\mathbb{C}^{2m} \ni (x_1, \dots, x_m, \xi_1, \dots, \xi_m) \mapsto (x_1, \dots, x_m) \in \mathbb{C}^m$. In order to compute the singular locus, we need to compute the solution $(x_1, \dots, x_m, \xi_1, \dots, \xi_m)$ with $(\xi_1, \dots, \xi_m) \neq (0, \dots, 0)$ for

$$L_1^B = 0, \dots, L_m^B = 0$$

i.e.,

$$x_i \xi_i = 0 \quad \text{or} \quad x_i(1 - x_i) \xi_i + \sum_{1 \leq k \leq m, k \neq i} x_k \xi_k = 0 \quad (i = 1, \dots, m). \quad (2)$$

The singular locus is the projection by π of these solution. We can rewrite the equation (2) as

$$x_i(1 - \varepsilon_i x_i) \xi_i + \sum_{1 \leq k \leq m, k \neq i} \varepsilon_i x_k \xi_k = 0 \quad (i = 1, \dots, m, \varepsilon_i \in \{0, 1\}). \quad (3)$$

We fix an $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$. The equation (3) is

$$\begin{pmatrix} x_1(1 - \varepsilon_1 x_1) & \varepsilon_1 x_2 & \cdots & \varepsilon_1 x_m \\ \varepsilon_2 x_1 & x_2(1 - \varepsilon_2 x_2) & \cdots & \varepsilon_2 x_m \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_m x_1 & \varepsilon_m x_2 & \cdots & x_m(1 - \varepsilon_m x_m) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We denote the coefficient matrix by A_ε . The equation (3) has a solution $(x_1, \dots, x_m, \xi_1, \dots, \xi_m)$ with $(\xi_1, \dots, \xi_m) \neq (0, \dots, 0)$ if and only if $\det(A_\varepsilon) = 0$ holds. The defining polynomial for the singular locus is $\prod_{\varepsilon \in \{0, 1\}^m} \det(A_\varepsilon)$.

We compute $\prod_{\varepsilon \in \{0, 1\}^m} \det(A_\varepsilon)$. We set

$$f_m(x_1, \dots, x_m; \varepsilon_1, \dots, \varepsilon_m) = \det \begin{pmatrix} 1 - \varepsilon_1 x_1 & \varepsilon_1 & \cdots & \varepsilon_1 \\ \varepsilon_2 & 1 - \varepsilon_2 x_2 & \cdots & \varepsilon_2 \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_m & \varepsilon_m & \cdots & 1 - \varepsilon_m x_m \end{pmatrix}.$$

Then, the relation $\det(A_\varepsilon) = x_1 \cdots x_m f_m(x_1, \dots, x_m; \varepsilon_1, \dots, \varepsilon_m)$ holds. We have

$$\begin{aligned} f_m(x_1, \dots, x_m; 0, \varepsilon_2, \dots, \varepsilon_m) &= f_{m-1}(x_2, \dots, x_m; \varepsilon_2, \dots, \varepsilon_m), \\ f_m(x_1, \dots, x_m; \varepsilon_1, 0, \varepsilon_3, \dots, \varepsilon_m) &= f_{m-1}(x_1, x_3, \dots, x_m; \varepsilon_1, \varepsilon_3, \dots, \varepsilon_m), \\ &\vdots \\ f_m(x_1, \dots, x_m; \varepsilon_1, \dots, \varepsilon_{m-1}, 0) &= f_{m-1}(x_1, \dots, x_{m-1}; \varepsilon_1, \dots, \varepsilon_{m-1}), \\ f_m(x_1, \dots, x_m; 1, \dots, 1) &= (-1)^{m-1} ((1 - x_1)x_2 \cdots x_m + x_1 x_3 \cdots x_m + \cdots \\ &\quad + x_1 \cdots x_{m-1}). \end{aligned}$$

By using these relation, we have

$$\begin{aligned}
& \prod_{\varepsilon \in \{0,1\}^m} f_m(x_1, \dots, x_m; \varepsilon_1, \dots, \varepsilon_m) \\
&= \prod_{\#\{1 \leq i \leq m | \varepsilon_i \neq 0\}=1} f_m \prod_{\#\{1 \leq i \leq m | \varepsilon_i \neq 0\}=2} f_m \cdots \prod_{\#\{1 \leq i \leq m | \varepsilon_i \neq 0\}=m} f_m \\
&= \prod_{1 \leq i_1 \leq m} f_1(x_{i_1}; 1) \prod_{1 \leq i_1 < i_2 \leq m} f_2(x_{i_1}, x_{i_2}; 1, 1) \cdots f_m(x_1, \dots, x_m; 1, \dots, 1) \\
&= \prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (x_{i_1} x_{i_2} - x_{i_1} - x_{i_2}) \cdots \\
&\quad (-1)^{m-1} (-x_1 x_2 \cdots x_m + x_2 \cdots x_m + \cdots + x_1 \cdots x_{m-1}).
\end{aligned}$$

These gives the following conclusion.

Theorem 8. *The singular locus of F_B is*

$$\begin{aligned}
\text{Sing}(I_B(m)) = & \mathbf{V}(x_1 \cdots x_m \prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (x_{i_1} x_{i_2} - x_{i_1} - x_{i_2}) \cdots \\
& (x_1 x_2 \cdots x_m - x_2 \cdots x_m - \cdots - x_1 \cdots x_{m-1})).
\end{aligned}$$

3.2 Singular locus of Lauricella's system $I_A(m)$

We compute the singular locus of Lauricella's system $I_A(m)$. In this case, the computation is not straightforward as the case of $I_B(m)$ and we need a Gröbner basis in the ring \mathcal{D} . We define a weight vector $(\mathbf{0}, \mathbf{1}) = (0, \dots, 0, 1, \dots, 1) \in \mathbb{Z}^{2m}$. The $(\mathbf{0}, \mathbf{1})$ initial form of ℓ_i^A is

$$\text{in}_{(\mathbf{0}, \mathbf{1})}(\ell_i^A) = x_i \xi_i \left(x_i \xi_i - x_i \sum_{1 \leq j \leq m} x_j \xi_j \right).$$

We denote the initial form by L_i^A . Since L_i^A is an element in the $(\mathbf{0}, \mathbf{1})$ initial form ideal $\text{in}_{(\mathbf{0}, \mathbf{1})}(I_A(m))$, we have

$$\langle L_1^A, \dots, L_m^A \rangle \subset \text{in}_{(\mathbf{0}, \mathbf{1})}(I_A(m)).$$

It holds that

$$\mathbf{V}(L_1^A, \dots, L_m^A) \supset \text{Ch}(I_A(m)).$$

For the singular locus, we have

$$\pi(\mathbf{V}(L_1^A, \dots, L_m^A) \setminus \{\xi_1 = \cdots = \xi_m = 0\}) \supset \text{Sing}(I_A(m)).$$

We compute $\pi(\mathbf{V}(L_1^A, \dots, L_m^A) \setminus \{\xi_1 = \cdots = \xi_m = 0\})$. In the analogous way as $I_B(m)$, we compute the solutions $(x_1, \dots, x_m, \xi_1, \dots, \xi_m)$ with $(\xi_1, \dots, \xi_m) \neq (0, \dots, 0)$ for

$$L_1^A = 0, \dots, L_m^A = 0$$

i.e.,

$$x_i \xi_i = 0 \text{ or } x_i \xi_i - x_i \sum_{1 \leq k \leq m} x_k \xi_k = 0 \quad (i = 1, \dots, m). \quad (4)$$

The projection by π of these solutions is $\pi(\mathbf{V}(L_1^A, \dots, L_m^A) \setminus \{\xi_1 = \dots = \xi_m = 0\})$. The equation (4) is rewritten as

$$x_i \xi_i - x_i \varepsilon_i \sum_{1 \leq k \leq m} x_k \xi_k = 0 \quad (i = 1, \dots, m, \varepsilon_i \in \{0, 1\}). \quad (5)$$

We fix an $\varepsilon = (\varepsilon_1, \dots, \varepsilon_m) \in \{0, 1\}^m$. The equation (5) is

$$\begin{pmatrix} x_1(1 - \varepsilon_1 x_1) & -\varepsilon_1 x_1 x_2 & \cdots & -\varepsilon_1 x_1 x_m \\ -\varepsilon_2 x_1 x_2 & x_2(1 - \varepsilon_2 x_2) & \cdots & -\varepsilon_2 x_2 x_m \\ \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_m x_1 x_m & -\varepsilon_m x_2 x_m & \cdots & x_m(1 - \varepsilon_m x_m) \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_m \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We denote the coefficient matrix by B_ε . The equation (5) has a solution $(x_1, \dots, x_m, \xi_1, \dots, \xi_m)$ with $(\xi_1, \dots, \xi_m) \neq (0, \dots, 0)$ if and only if $\det(B_\varepsilon) = 0$ holds. So we have

$$\pi(\mathbf{V}(L_1^A, \dots, L_m^A) \setminus \{\xi_1 = \dots = \xi_m = 0\}) = \prod_{\varepsilon \in \{0, 1\}^m} \det(B_\varepsilon).$$

We compute the determinant.

$$\begin{aligned} \det(B_\varepsilon) &= x_1 \cdots x_m \det \begin{pmatrix} 1 - \varepsilon_1 x_1 & -\varepsilon_1 x_1 & \cdots & -\varepsilon_1 x_1 \\ -\varepsilon_2 x_2 & 1 - \varepsilon_2 x_2 & \cdots & -\varepsilon_2 x_2 \\ \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_m x_m & -\varepsilon_m x_m & \cdots & 1 - \varepsilon_m x_m \end{pmatrix} \\ &= x_1 \cdots x_m \det \begin{pmatrix} 1 - \varepsilon_1 x_1 & -1 & -1 & \cdots & -1 \\ -\varepsilon_2 x_2 & 1 & 0 & \cdots & 0 \\ -\varepsilon_3 x_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\varepsilon_m x_m & 0 & 0 & \cdots & 1 \end{pmatrix} \\ &= x_1 \cdots x_m (1 - \varepsilon_1 x_1 - \varepsilon_2 x_2 - \cdots - \varepsilon_m x_m) \end{aligned}$$

So we obtain

$$\begin{aligned} \prod_{\varepsilon \in \{0, 1\}^m} \det(A_\varepsilon) &= x_1^{2^m} \cdots x_m^{2^m} \prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (1 - x_{i_1} - x_{i_2}) \cdots \\ &\quad (1 - x_1 - \cdots - x_m). \end{aligned}$$

This equation is the defining polynomial of $\pi(\mathbf{V}(L_1^A, \dots, L_m^A) \setminus \{\xi_1 = \dots = \xi_m = 0\})$. We have

$$\begin{aligned} \text{Sing}(I_A(m)) &\subset \mathbf{V}(x_1 \cdots x_m \prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (1 - x_{i_1} - x_{i_2}) \cdots \\ &\quad (1 - x_1 - \cdots - x_m)) \end{aligned}$$

We prove the reverse inclusion. We compute the singular locus in an open neighborhood of the origin. By Theorem 4, a Gröbner basis of $\mathcal{I}_A(m)$ with respect to $<_{(\mathbf{0}, \mathbf{1})}'$ is $\{\ell_1^A, \dots, \ell_m^A\}$. For a \mathcal{D} ideal \mathcal{I} , the $(\mathbf{0}, \mathbf{1})$ local initial form ideal is defined by the $\mathbb{C}\{x_1, \dots, x_m\}[\xi_1, \dots, \xi_m]$ ideal

$$\text{in}_{(\mathbf{0}, \mathbf{1})}(\mathcal{I}) = \langle \text{in}_{(\mathbf{0}, \mathbf{1})}(P) \mid P \in \mathcal{I} \rangle.$$

By the property of a Gröbner basis with respect to $<_{(\mathbf{0}, \mathbf{1})}'$ [7, Theorem in Section 2], the $(\mathbf{0}, \mathbf{1})$ local initial form ideal $\text{in}_{(\mathbf{0}, \mathbf{1})}(\mathcal{I}_A(m))$ are generated by L_1^A, \dots, L_m^A . By the theorem in [6, Th 4.1.], we obtain the following fact.

Proposition 9. *For the characteristic variety for the D ideal $I_A(m)$, there exists an open neighborhood U of $\{x_1 = \dots = x_m = 0\}$ in \mathbb{C}^{2m} such that $\text{Ch}(I_A(m)) \cap U = \mathbf{V}(L_1^A, \dots, L_m^A) \cap U$.*

We compute the solutions $(x_1, \dots, x_m, \xi_1, \dots, \xi_m)$ with $(\xi_1, \dots, \xi_m) \neq (0, \dots, 0)$ for $L_1^A = 0, \dots, L_m^A = 0$. The projection by π of these solutions in an open neighborhood of the origin is the singular locus in the open neighborhood. The equations $L_i^A = 0$ ($i = 1, \dots, m$) have a solution $(x_1, \dots, x_m, \xi_1, \dots, \xi_m)$ with $(\xi_1, \dots, \xi_m) \neq (0, \dots, 0)$ if and only if

$$\prod_{\varepsilon \in \{0, 1\}^m} \det(A_\varepsilon) = x_1^{2^m} \cdots x_m^{2^m} \prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (1 - x_{i_1} - x_{i_2}) \cdots (1 - x_1 - \cdots - x_m)$$

holds. Since we are considering an open neighborhood of the origin, the factors

$$1 - x_{i_1}, 1 - x_{i_1} - x_{i_2}, \dots, 1 - x_1 - \cdots - x_m$$

are not 0. We have

$$\text{Sing}(I_A(m)) \cap W = \mathbf{V}(x_1 \cdots x_m) \cap W,$$

where W is an open neighborhood of the origin in \mathbb{C}^m .

Next, we compute the singular locus $\text{Sing}(I_A(m))$ in the complex torus $(\mathbb{C}^*)^m$. We assume $x_i \neq 0$ ($i = 1, \dots, m$) i.e., $(x_1, \dots, x_m) \in (\mathbb{C}^*)^m$. We apply the change of coordinates $X_i = \frac{1}{x_i}$ ($i = 1, \dots, m$). By the change of coordinates, the differential operator ℓ_i^A changes to

$$p_i^A = X_i \theta_{X_i} (-\theta_{X_i} + c_i - 1) + (\theta_{X_i} - a)(\theta_{X_i} - b_i).$$

We set the D ideal

$$I_A(m)' = D \cdot \{p_1^A, \dots, p_m^A\}.$$

Here, we set $D = \mathbb{C}[X_1, \dots, X_m] \langle \partial_{X_1}, \dots, \partial_{X_m} \rangle$, i.e., we change the variables x_i, ∂_i for X_i, ∂_{X_i} .

Proposition 10. *We use the term order $<_{(\mathbf{0}, \mathbf{1})}$ defined in Theorem 1. Here, we change the variable x_i, ∂_i for X_i, ∂_{X_i} . The set $\{p_1^A, \dots, p_m^A\}$ is a Gröbner basis for the D ideal $I_A(m)'$ with respect to $<_{(\mathbf{0}, \mathbf{1})}$.*

We can analogously prove the proposition as Theorem 1. We compute the singular locus of $\text{Sing}(I_A(m)')$. By the property of a Gröbner basis with respect to $<_{(\mathbf{0}, \mathbf{1})}$, the $(\mathbf{0}, \mathbf{1})$ initial form ideal $\text{in}_{(\mathbf{0}, \mathbf{1})}(I_A(m)')$ is generated by $\text{in}_{(\mathbf{0}, \mathbf{1})}(p_1^A), \dots, \text{in}_{(\mathbf{0}, \mathbf{1})}(p_m^A)$. The $(\mathbf{0}, \mathbf{1})$ initial form of p_i^A is

$$\text{in}_{(\mathbf{0}, \mathbf{1})}(p_i^A) = X_i \xi_{X_i} \left(X_i(1 - X_i) \xi_{X_i} + \sum_{1 \leq j \leq m, j \neq i} X_j \xi_{X_j} \right).$$

We denote it by P_i^A . Since P_i^A is equal to L_i^B , $\text{in}_{(\mathbf{0}, \mathbf{1})}(I_A(m)')$ is equal to $\text{in}_{(\mathbf{0}, \mathbf{1})}(I_B(m))$. By the result of $\text{Sing}(I_B(m))$, we have

$$\begin{aligned} \text{Sing}(I_A(m)') = & \mathbf{V}(X_1 \cdots X_m \prod_{1 \leq i_1 \leq m} (1 - X_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (X_{i_1} X_{i_2} - X_{i_1} - X_{i_2}) \\ & \cdots (X_1 X_2 \cdots X_m - X_2 \cdots X_m - \cdots - X_1 \cdots X_{m-1})). \end{aligned}$$

The relation $X_i \neq 0$ holds and we apply the change of coordinates $X_i = \frac{1}{x_i}$, we have

$$\begin{aligned} \text{Sing}(I_A(m)) \cap (\mathbb{C}^*)^m = & \mathbf{V} \left(\prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (1 - x_{i_1} - x_{i_2}) \cdots \right. \\ & \left. (1 - x_1 - \cdots - x_m) \right). \end{aligned}$$

Since $\text{Sing}(I_A(m)) \cap W = \mathbf{V}(x_1 \cdots x_m) \cap W$ (W is an open neighborhood of the origin) holds, we have

$$\begin{aligned} \text{Sing}(I_A(m)) \supset & \mathbf{V}(x_1 \cdots x_m \prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (1 - x_{i_1} - x_{i_2}) \cdots \\ & (1 - x_1 - \cdots - x_m)). \end{aligned}$$

We can prove the reverse inclusion.

Theorem 11. *The singular locus of F_A is*

$$\begin{aligned} \text{Sing}(I_A(m)) = & \mathbf{V}(x_1 \cdots x_m \prod_{1 \leq i_1 \leq m} (1 - x_{i_1}) \prod_{1 \leq i_1 < i_2 \leq m} (1 - x_{i_1} - x_{i_2}) \cdots \\ & (1 - x_1 - \cdots - x_m)). \end{aligned}$$

Remark 12. *In the analogous way as Lauricella's system $I_A(m)$, we can also determine the singular locus of Lauricella's system $I_C(m)$. Hattori and Takayama [4] determined the singular locus of the system $I_C(m)$, but our method is simpler than their method.*

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